1 Vectors and their Operations

Vectors are everywhere, in pretty much every STEM field, economics, and even graphic design, but the true definition of a vector is a little too complicated—at least, assuming no experience with the purer side of math—for me to give you as a first chapter.

Unfortunately, it seems I must do what I hate most: give only half of a definition at a time. For now, we'll start with something more intuitive, the most well-known example of a vector (which, for convenience, will be referenced simply as a vector).

In this chapter, we'll cover the basics: how vectors can be added, scaled, and subtracted, and we'll explore operations like the dot and cross products. We'll also touch on how vectors can be broken down into components and their magnitudes, all without getting too abstract just yet. By the end of this chapter, you'll have a solid grasp on how to work with vectors in the most straightforward ways. So, let's dive in!

1.1 Definition and properties

You might be thinking: *what exactly is a vector*? (If you've already heard of them, I'd still encourage reading this chapter to refresh yourself.) Well, to understand what a vector is, let's go back to something we're more familiar with: the real numbers.

The set of real numbers (henceforth denoted as $\mathbb{R}$) contains elements such as 3, -20, $\frac{7}{2}$, 0.00002307, and $\sqrt{2}$. Now, what do they have in common? Think about the notable things a number is defined by.

One thing that might come to mind is their distance from zero, otherwise known as the absolute value of that number. This gives us a sense of how large that number is. 10 is greater than 9, which is greater than 8.9, which is greater than $\sqrt[3]{8.9}$, and so on. A loss of \$10.01 is worse than a loss of \$10 because its absolute value, its magnitude, is greater.

Another thing that all real numbers have is a sign: $+$ or $-$ (well, except for zero). The sign of a number tells us which size of zero it is on, the difference between left and right, the direction of the number.

Of course, there are a lot of other things we could say (even vs. odd, rational vs. irrational), but let's stick to magnitude and direction. While there are infinitely many magnitudes a number can have in $\mathbb{R}$, there are really only two directions, confining us to a single dimension. Try and remember how we went from a single direction to two.

That's right, the x-y plane. Now, we're working with points like $(3, 2)$, $(7.07, -\ln(5))$, etc. Our zero in one dimension now becomes the point $(0, 0)$, called the origin, and we can define the magnitude of a point as its distance from the origin, just like the absolute value sign gives us the distance from our number to zero.

Redefining direction is a little trickier. Our first instinct might be to keep using positive and negative signs. A number was positive if it was to the right of zero on a number line, and negative if it was on the left. On an x-y plane, there are actually two zeroes that a point can be above/to the left of or below/to the right of. Two zeroes times two choices for placement gives us four quadrants (top-right, top-left, bottom-left, and bottom-right relative to the origin) that we can put our points in.

However, there's a problem with calling those four quadrants *directions*. On a line (one dimension), you can really only go forwards and backwards. The amount by which you do so may differ, but that's part of the magnitude and not the direction. However, on a plane (two dimensions), there are a lot more ways you can move than simply up, down, left, or right.

It's not too hard to demonstrate this. Convince yourself that moving in a flat room, so long as you don't jump, confines you to a two-dimensional plane. Take wherever you're currently standing as the origin of that plane, and the direction you're looking in as the positive x-axis. Turning 180 degrees in either direction will then have you facing the negative x-axis. From there, turning 90 degrees in both directions will give you the positive and negative y-axes. These can be called up, down, left, and right. However, it shouldn't be hard to move in a direction that isn't simply one of those four.

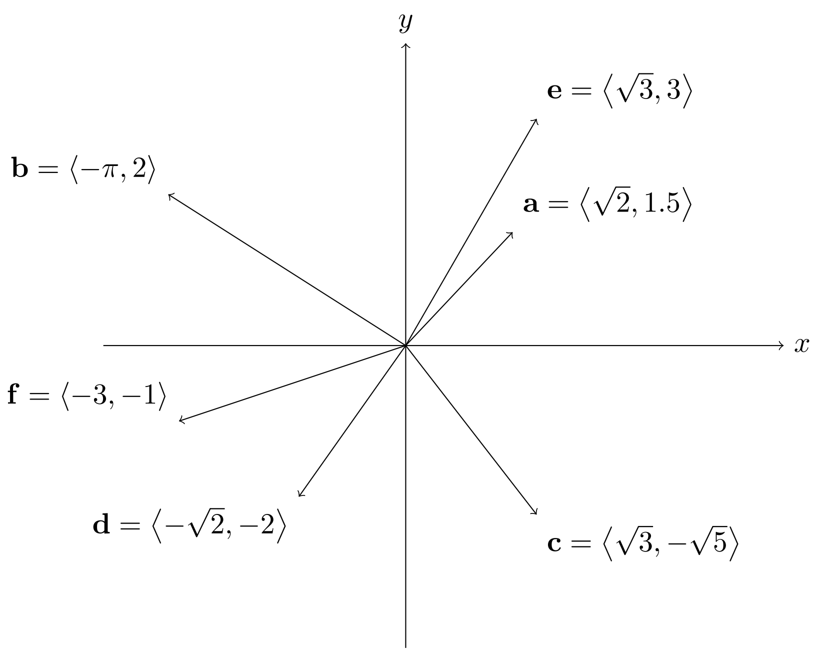
We can see now that quadrants, defined by positive and negative signs, won't be enough to adequately define the direction of a point relative to the origin. They simply aren't precise enough. What might be a better way to denote direction? (HINT: I used it in the above paragraph.)

If you were thinking degrees (radians are also acceptable, but I won't use those for now), you're correct. Let's say that anything on the positive x-axis has a direction of zero degrees. We know that the x and y-axes are perpendicular, so anything on the positive y-axis will have a direction of 90 degrees. Similarly, the negative x and y-axes will have directions of 180 and 270 degrees. Then, no matter how much we spin on the origin, our direction can always be represented as some amount of degrees.

Now, we finally have enough background to define a vector (or at least, the most basic example of one).

Definition 1.1.1. A Euclidean vector in n dimensions is a mathematical object (denoted $\langle x_1, x_2, ..., x_n \rangle$) with magnitude and direction.

As the chapter progresses, we'll be adding more to this definition. For now though, let's see some examples of 2D vectors (such vectors are said to be in $\mathbb{R}^2$, basically saying that it has two components and both of them must be real numbers).



As you can see, a Euclidean vector $\langle x, y \rangle$ is basically an arrow going from the origin to the point (x, y). The magnitude of a vector is therefore the length of the arrow, and it's direction is the angle between it and the positive x-axis (it's standard to measure angles in this way, and that's how I'll be doing so unless explicitly stated otherwise). To find the magnitude, simply use the distance formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In this case, the point (x_1, y_1) is the origin and therefore the distance formula can be simplified to the square root of the sum of the squares of the x and y-coordinates.

To find the direction of a 2D vector, simply remember trigonometry and that the slope of a line (rise over run) equals the tangent of the angle the line makes with the x-axis. Therefore, since every vector starts at the origin, the slope is simply the y-component over the x-component, and we have the following:

$$m = \tan(\theta)$$

$$\frac{y}{x} = \tan(\theta)$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

Finding the magnitude of vectors in higher dimensions (like $\mathbb{R}^3$ and $\mathbb{R}^4$) is pretty simple: simply add the squares of all the components, then take the square root. In n dimensions, the magnitude of a vector $v = \langle x_1, x_2, ..., x_n \rangle$ is as follows:

$$d = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$$

Once again, direction of vectors in higher dimensions isn't that simple, represented by special unit vectors, which will be explained later in the chapter.

On the other hand, it's pretty easy to do vectors in one dimension. The magnitude of some vector $v = \langle x \rangle$ is just $\sqrt{x^2} = |x|$, and it can only point in the positive or negative direction. This sounds pretty familiar. That's because it acts exactly like the real numbers, we're just looking at it in a different context.

Vectors, like numbers, aren't tied to any specific units. They can represent things like forces, velocities, and accelerations just like real numbers can. However, real numbers can only give us the magnitude of whatever it is they're quantifying. I can say I'm pushing a box with a force of 4 Newtons, but that doesn't tell you what direction I'm pushing the box. It's when I say I'm pushing that box with a force of 4 Newtons 30° relative to true north, when I'm giving you a vector representing the force, that you can know both magnitude and direction.

That brings us to the end of this section. In the next one, we'll define common operations like addition and subtraction in the context of vectors, and also look at how to scale them.